Computational Inquiry in Undergraduate Math Courses

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Modern computational tools can help students explore and discover mathematical ideas for themselves.

Are we teaching students to use these tools?

How do our students experience the process of mathematical exploration and discovery?

Computation can:

Reveal patterns

Suggest what might be true

Test conjectures

Suggest approaches for proof

Make advanced math more accessible

MATH 242 Modern Computational Mathematics

St. Olaf College

Focuses on the "how" of mathematics, not the "what"

Teaches students to ask mathematical questions, perform mathematical experiments, and formulate precise conjectures

A "transition course" for the math major

Prerequisite: linear algebra

Is not: numerical analysis, computer science, or data science

Topic 1: Collatz sequences

Start with a positive integer n. Repeatedly apply the function

$$\mathrm{Col}(n) = \begin{cases} 3n+1 & \text{if } n \text{ is odd,} \\ n/2 & \text{if } n \text{ is even.} \end{cases}$$

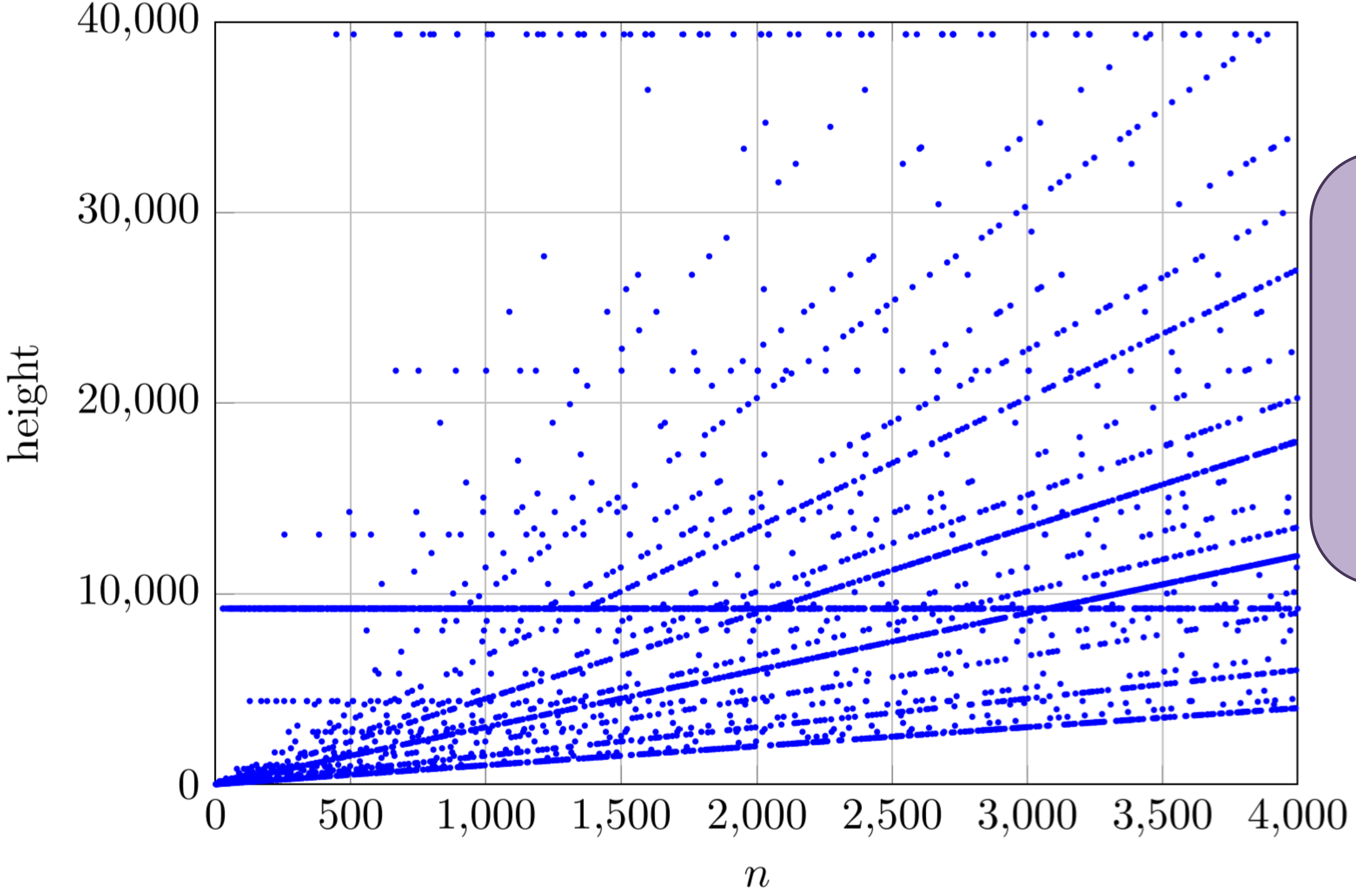
Examples:

$5 \to 16 \to 8 \to \underline{4 \to 2 \to 1 \to 4 \to 2 \to 1}$

$7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16$
$\to 8 \to \underline{4 \to 2 \to 1}$

Collatz conjecture: Starting with any positive integer n, the sequence of Collatz iterates eventually reaches 1.

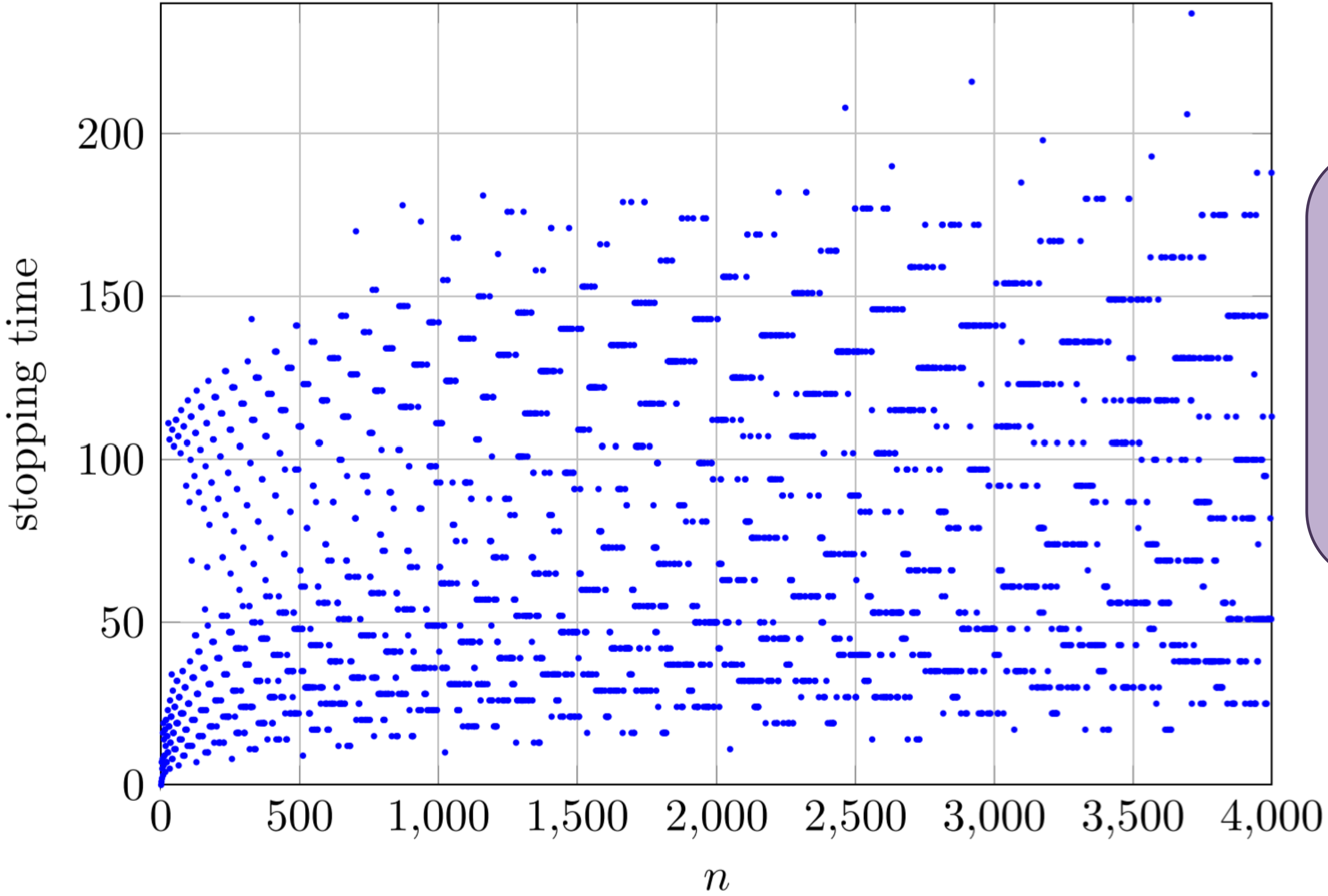
Heights of Collatz sequences



What do you observe?

What do you wonder?

Collatz stopping times



What do you observe?

What do you wonder?

Characteristics of a good mathematical question

1. The question is interesting to you.
2. You don’t already know the answer to the question.
3. You haven’t already seen the question before, or at least not exactly.
4. You can begin to make computational investigations to shed light on the question.

from Barry Mazur and William Stein, *Prime Numbers and the Riemann Hypothesis*

Topic 2: Percolation Theory

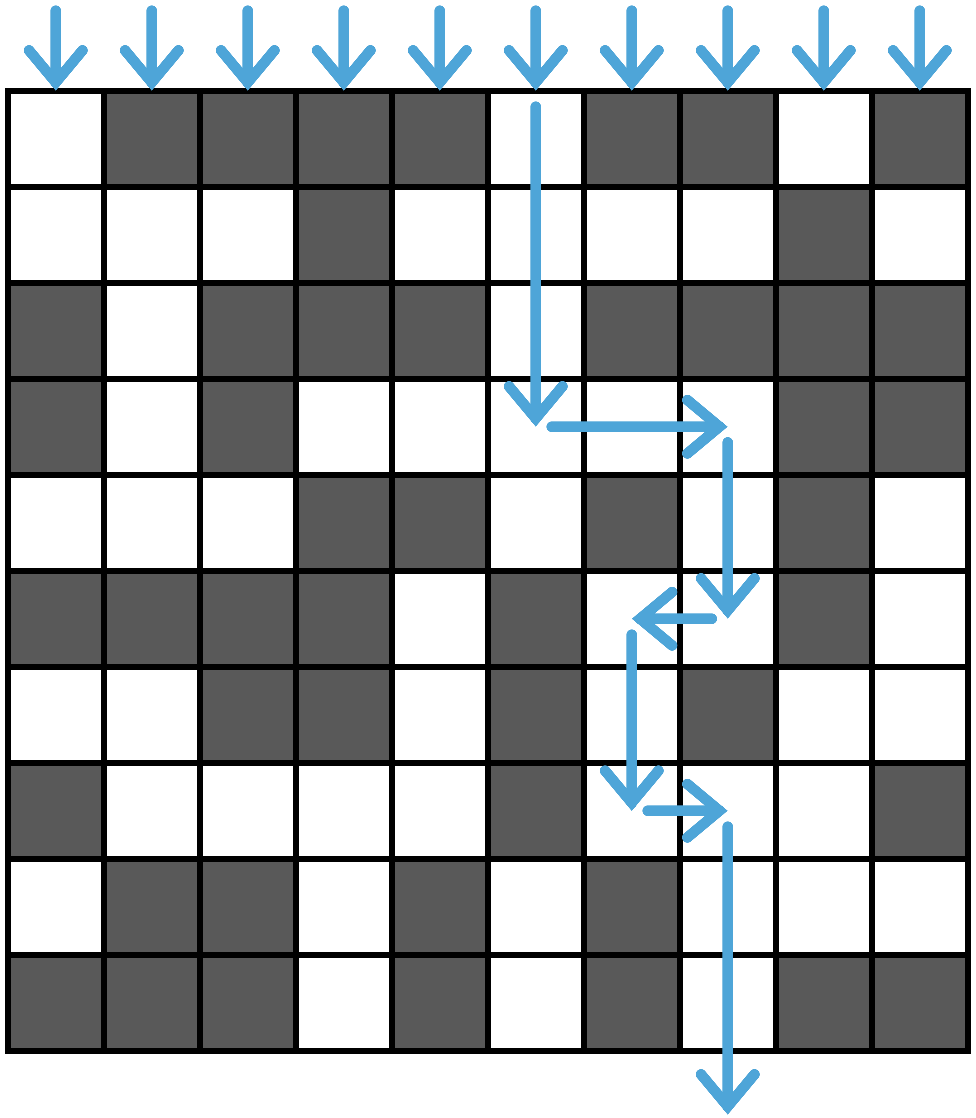
Consider an $n \times n$ grid of squares.

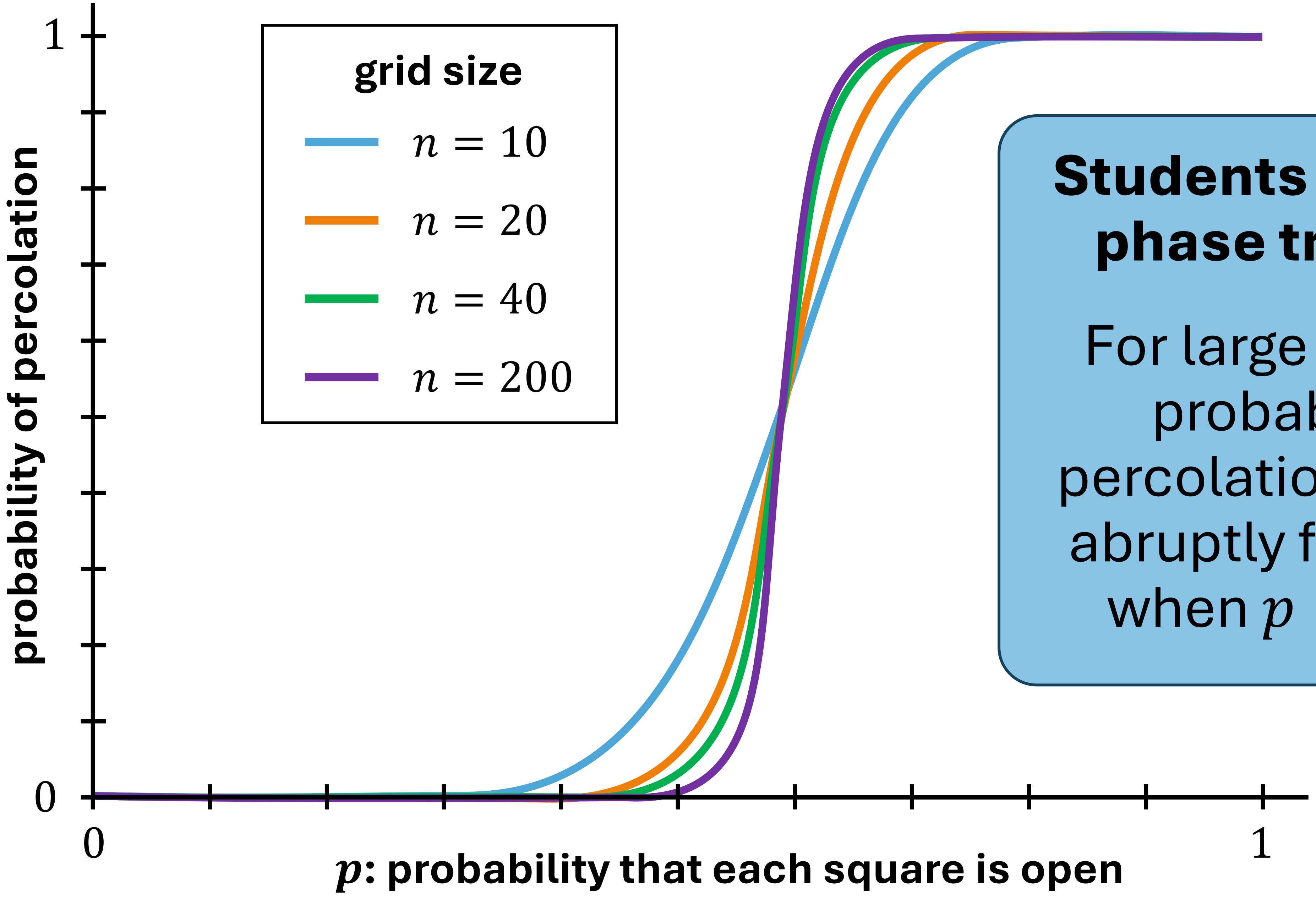
Each square is "open" with probability p and "closed" with probability $1 - p$.

Water is poured on the top and flows through adjacent open squares.

If water can flow from top to bottom, then we say a *percolation path* exists.

What is the probability that a percolation path exists? How does this depend on n and p?





Students observe a phase transition

For large grids, the probability of percolation changes abruptly from 0 to 1 when $p \approx 0.595$.

Computational skill in probabilistic simulation allows students to gain insight into a huge variety of questions.

These questions have forms such as:

- *What is the probability that…*
- *On average, how long does it take to…*

For these questions, an exact answer may be far out of students' reach, but a simulation can provide good intuition and estimates of the answer.

Topic 3: Primes and the Riemann Hypothesis

Riemann zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } \mathrm{Re}(s) > 1$$

Riemann spectrum: sequence of imaginary parts of nontrivial zeros of $\zeta(s)$

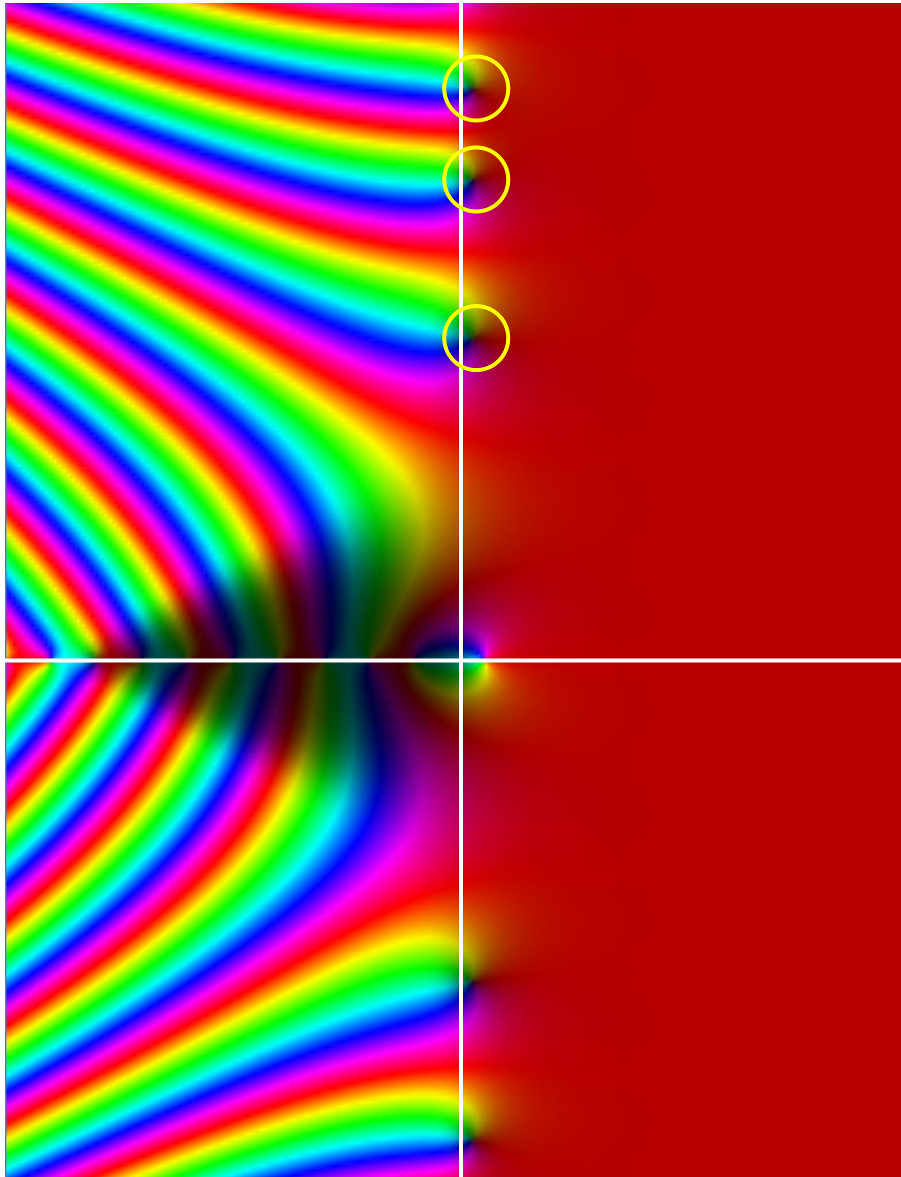
$$\theta_1 = 14.134725\ldots$$

$$\theta_2 = 21.022039\ldots$$

$$\theta_3 = 25.010857\ldots$$

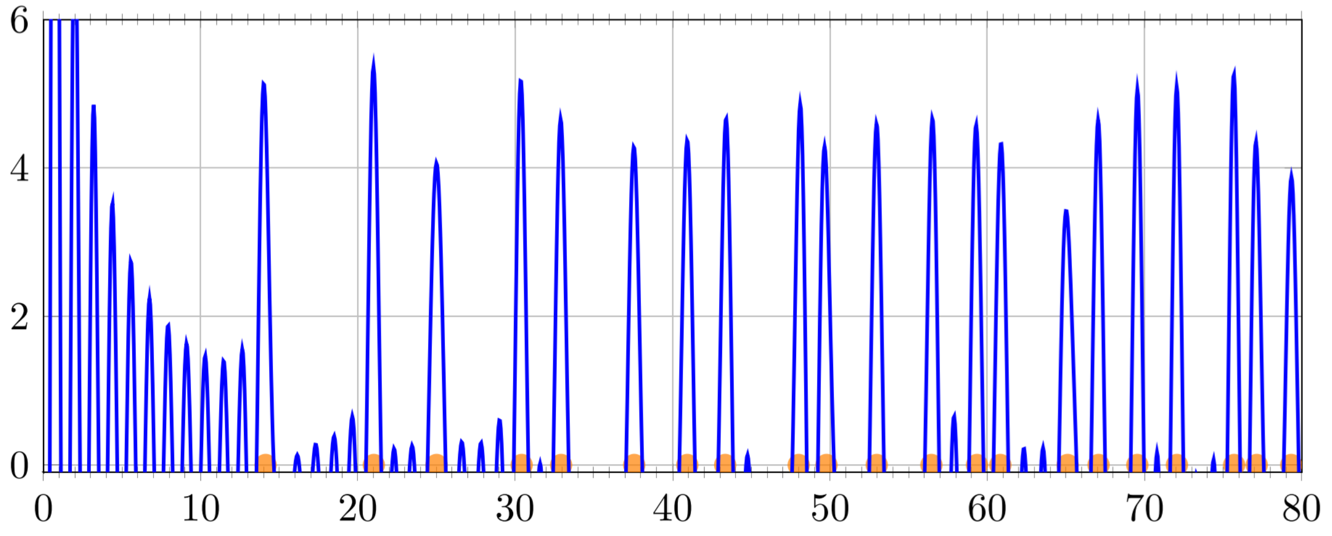
$$\theta_4 = 30.424876\ldots$$

$$\vdots$$

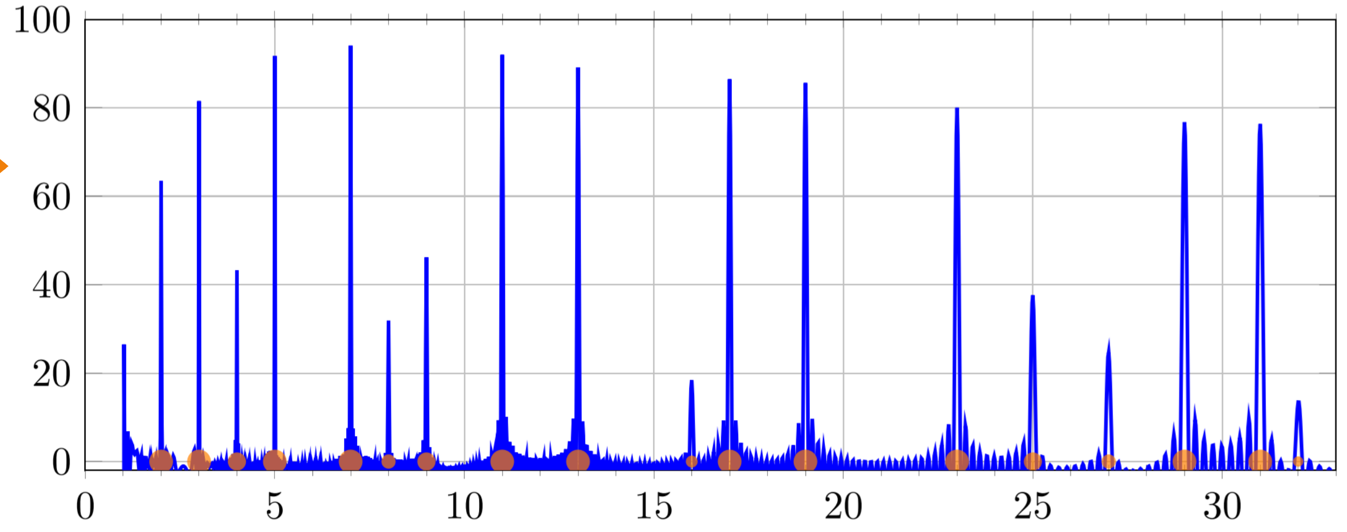


How do the primes relate to the Riemann spectrum?

$$F(\theta) = -\sum_{\substack{\text{prime} \\ \text{powers } p^k}} \frac{\ln(p)}{p^{k/2}} \cos\left(\theta \ln\left(p^k\right)\right)$$



$$f(s) = -\sum_{k=1}^{\infty} \cos(\ln(s)\, \theta_k)$$



The primes and the Riemann spectrum are dual sequences!

Computation highlights for students how the **notion of certainty** in mathematics differs from that in other scientific disciplines.

Nontrivial zeros of the Riemann zeta function have been computed up to a height of 10 trillion. This is an enormous amount of evidence in support of the hypothesis.

In other areas of science, this amount of evidence would lead to a scientific law, but in mathematics the Riemann hypothesis remains a conjecture.

Modern Computational Math

math242.mlwright.org

Modern Computational Math

Math 242 · Spring 2024

Welcome to Modern Computational Math! For grades, log into Moodle. If you need help, contact Prof. Wright.

Prof. Wright's office hours: Mon. 9–10am, Tues. 2–3pm, Wed. 11am–12pm, Thurs. 1–2pm, Fri. 2–3pm, and other times by appointment (in RMS 405)

Help sessions: Tuesdays 7:15–8:15pm, Thursdays 6–7pm, Sundays 6–7pm in Tomson 188

SCHEDULE | SYLLABUS | FILES | LINKS

Top | Today | Bottom | Challenge Problems

Do the following before the first class:

- Complete the Introductory Survey.
- Install Mathematica on your computer. If you've already installed Mathematica, open it up and check that your license key is still active. You might be prompted to upgrade to the most recent version. For assistance, see this IT Help Desk page.

Wednesday February 7 — Introduction; Mathematica basics

Intro Mathematica HW | in-class problems | notes and solutions | Introductory Survey | Computational Mathematics Text

Do the following before next class:

- Complete the Introductory Survey, if you haven't done so already.
- Read the Syllabus. Pay special attention to the grading information.
- Watch the video Hands-On Start to Mathematica by Wolfram.
- Read pages 1–10 of Computational Mathematics. Come to class prepa
- Archimedes's method for computing π.
- Complete the Intro Mathematica homework problems and submit your n
- Mathematica assignment on Moodle.

Friday February 9 — Computing π

Archimedes method HW | classwork | in-cl

Do the following before next class:

- Read the following pages about the Wolfram Language: Fractions & Deci
- Functions, Lists, Iterators, and Assignments.
- Modify the code from class to complete the Archimedes's Method practi
- upload your solutions to the Archimedes's Method assignment on Moo

Textbook in preparation

AMS/MAA | TEXTBOOKS

Exploring Mathematics from a Computational Perspective

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These slides: